

M303 Further Pure Mathematics
Exam 2 June 2017
**Unofficial Solutions - these have NOT been
checked by the module team. They will likely
contain mistakes and omissions but may be
helpful for revision purposes.**
(Student version)

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2018
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Caveat Lector

These solutions are not my solutions. They represent the work of students from the cohort of 2016/17 who can be rightly proud of the quality of the answers reproduced below. Of course, that is not to say that there is no room for improvement, but it still represents an excellent piece of collaborative student effort.

Nor are they, in any sense, official solutions. No member of the Module Team has played any part in the compilation of these answers.

I have chosen to exploit my position as editor in one small stylistic matter: I prefer to see ‘Abelian’ rather than ‘abelian’ and have used the capitalised version throughout.

Matters of style are, in essence, matters of opinion. I should point out that most algebraists do not use the capital, and that Book B is consistent with the practice of most of the group theorists I have known. I just happen to think they are misguided.

For those who care about such matters, no marks would be lost (nor, alas, gained) for writing ‘Abelian’ rather than ‘abelian’.

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Q 1.

(a)

(i) Applying the Euclidean Algorithm we have:

$$312 = 2 \times 143 + 26$$

$$143 = 5 \times 26 + 13$$

$$26 = 2 \times 13 + 0$$

So $\text{hcf}(143, 312) = 13$ (since this is the last non-zero remainder).

(ii) We seek a solution to the equation: $143s + 312t = 13$, so we use the equations above to express 13 as a product of 143 and 312 as follows:

$$\begin{aligned} 13 &= 143 - 5 \times 26 \\ &= 143 - 5 \times (312 - 2 \times 143) \\ &= -5 \times 312 + 11 \times 143 \end{aligned}$$

Therefore we have found one solution to this Linear Diophantine Equation with: $s = 11$ and $t = -5$

To check:

$$11 \times 143 = 1573$$

$$5 \times 312 = 1560$$

So $1573 - 1560 = 13$ as required.

Not needed, but useful.

(b) We use the Principle of Mathematical Induction (Handbook, p14). Let $P(n)$ be the statement:

$$\sum_{i=1}^n (3i - 2) = \frac{n(3n - 1)}{2}.$$

1. We now check the case $n = 1$ to form the basis for the induction:

For $n = 1$, LHS of $P(1) = 3 \times 1 - 2 = 1$, RHS of $P(1) = \frac{1 \times (3 - 1)}{2} = 1$.

Therefore LHS of $P(1) = \text{RHS of } P(1)$ so we can confirm that $P(1)$ is true.

2. Now, assume that $P(k)$ is true for some $k \in \mathbb{N}$, that is:

$$\sum_{i=1}^k (3i - 2) = \frac{k(3k - 1)}{2}$$

We want to deduce the truth of $P(k + 1)$. That is we want to show that:

$$\sum_{i=1}^{k+1} (3i - 2) = \frac{(k + 1)(3k + 2)}{2}$$

Starting with the LHS we have:

$$\begin{aligned}\sum_{i=1}^{k+1} (3i - 2) &= \sum_{i=1}^k (3i - 2) + [3(k + 1) - 2] \\ &= \frac{k(3k - 1)}{2} + 3k + 1 \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k + 1)(3k + 2)}{2}\end{aligned}$$

As required.

Therefore the truth of $P(k + 1)$ has been confirmed and the induction process is complete. So by Mathematical Induction we can conclude the truth of $P(n)$.

Q 2.

(a)

(i) $|G| = 24$.

(ii) 2

(iii) (0,1)

(iv) No, since $\text{lcm}(4, 6) = 12$, by Proposition 2.10, p32, HB, no element in G has order greater than 12, and as $12 < |G|$, G is not cyclic.

(b)

(i) We know that $sr^{-i} = r^i s$ (HB, p34). Thus:

$$r^6 sr^6 = r^6 (sr^{-6}) = r^6 r^6 s = s.$$

(ii) Clearly $r^6 s \neq e$. Also, using the previous part,

$$(r^6 s)^2 = r^6 sr^6 s = (r^6 sr^6) s = ss = e,$$

from which it follows that $r^6 s$ has order 2.

(iii) $H = \langle r \rangle = \{e, r, r^2, r^3, r^4, r^5, r^6, r^7, r^8, r^9, r^{10}, r^{11}\}$ is a subgroup of D_{12} of order 12. .

Full marks for $\langle r \rangle$

(iv) Yes, as it has index 2 in D_{12} .

Can't find HB ref.,
but see Ex. 3.3 on
p30 of Book B.

Q 3.

(a)

(i) Using Proposition 2.3 HB Page 17,

$$\tau(75) = (1+1)(2+1) = 2 \times 3 = 6.$$

(ii) By Proposition 1.7 HB Page 42,

$$\sigma(75) = (1+3)(1+5+5^2) = 4 \times 31 = 124.$$

(iii) By Alternative Formula below Proposition 4.2 HB Page 44,

$$\phi(75) = 3^0 \times 5^1 \times (3-1)(5-1) = 1 \times 5 \times 2 \times 4 = 40.$$

(b) Using Definition 2.1 HB Page 45 and a selection of Results from Pages 46 and 47 as noted in the solutions we have:

(i) $(2/199) = 1$, since $199 \equiv 7 \pmod{8}$. See Proposition 3.2, HB Pg 46.

(ii)

$$\begin{aligned} (179/199) &= (-20/199) && \text{since } 179 \equiv -20 \pmod{199} \text{ (Thm 2.2(a))} \\ &= (-1/199)(4/199)(5/199) && \text{Thm 2.2(c)} \\ &= -1 \times 1 \times (199/5) && \text{Thm 2.2(e) } (5 \equiv 1 \pmod{4}), \text{ Thm 2.2(b) and Thm 4.2 respectively} \\ &= -(4/5) && \text{since } 199 \equiv 4 \pmod{5}, \text{ (Thm 2.2(a))} \\ &= -1 && \text{Thm 2.2(b).} \end{aligned}$$

(c) N.B. Q3(c) contained a typo. We believe the correct expression for $f(x)$ should have been

$$f(x) = x^4 + x^3 + 16x^2 - 49x + 36$$

not

$$f(x) = x^4 + x^3 + 16x^2 - 49x + 34.$$

We therefore work with the corrected polynomial.

(i) If $f(x) = x^4 + x^3 + 16x^2 - 49x + 36$, then using binomial expansion we have

$$\begin{aligned} f(x+1) &= (x+1)^4 + (x+1)^3 + 16(x+1)^2 - 49(x+1) + 36 \\ &= x^4 + 4x^3 + 6x^2 + 4x + 1 + x^3 + 3x^2 + 3x + 1 + 16x^2 + 32x + 16 - 49x - 49 + 36 \\ &= x^4 + 5x^3 + 25x^2 - 10x + 5. \end{aligned}$$

Students who worked with

$$f(x) = x^4 + x^3 + 16x^2 - 49x + 34.$$

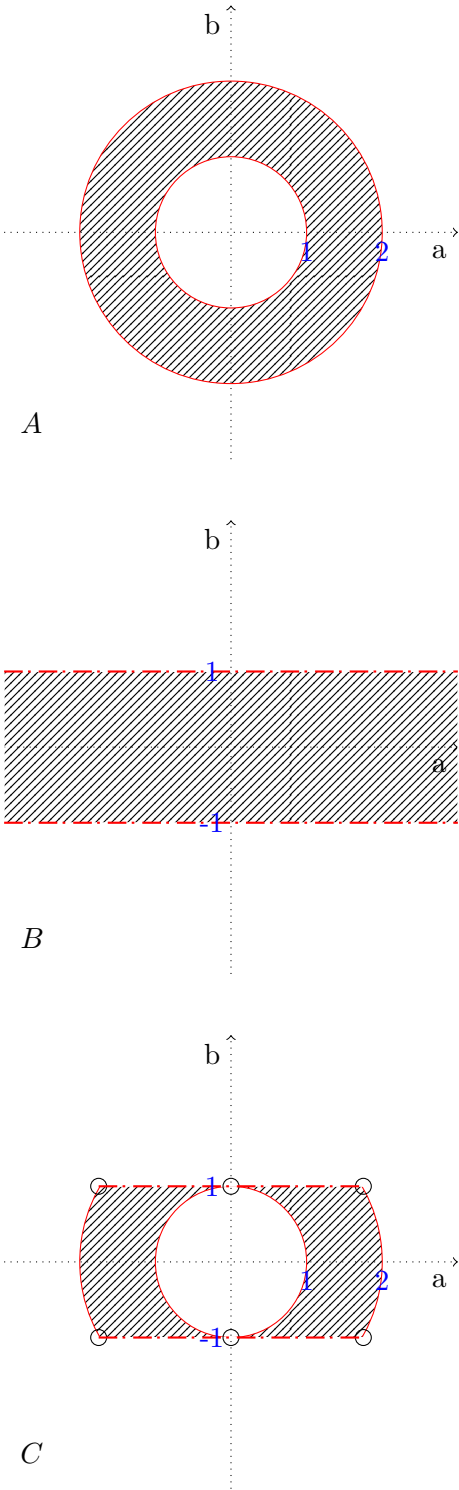
would get $f(x+1) = x^4 + 5x^3 + 25x^2 - 10x + 3$.(ii) We have $f(x+1) = x^4 + 5x^3 + 25x^2 - 10x + 5$.

Using Eisenstein's Criterion from HB p. 52,
since $f(x+1)$ is a polynomial in $\mathbb{Z}[x]$ and the prime $5 \in \mathbb{Z}$ is such that
(a) 5 divides each of the coefficients except for the leading one, and
(b) $5^2 = 25$ does not divide the coefficient of x^0 , which is 5,
then $f(x+1)$ is irreducible in $\mathbb{Q}[x]$.

Finally using Lemma 4.17 from HB p. 52,
since $f(x)$ is a polynomial in $\mathbb{Z}[x]$, and
since $f(x+1) = f(x - (-1))$ is irreducible in $\mathbb{Q}[x]$, and $-1 \in \mathbb{Q}$,
then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Q 4.

(a)



(b) For the Euclidean metric:

$$\begin{aligned}\text{Cl}(C) &= \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4, |b| \leq 1\}, \\ \text{Int}(C) &= \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 4, |b| < 1\}, \\ \text{Bd}(C) &= \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 4, |b| \leq 1\} \cup \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\} \\ &\quad \cup \{(a, b) \in \mathbb{R}^2 : a^2 \leq 3, |b| = 1\}\end{aligned}$$

(c) Let

$$U = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4, a < 0, |b| < 1\}$$

and

$$V = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4, a > 0, |b| < 1\}$$

Then $U \cup V = C$ and as

$$\text{Cl}(U) = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4, a \leq 0, |b| \leq 1\}$$

and

$$\text{Cl}(V) = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 \leq 4, a \geq 0, |b| \leq 1\}$$

$\text{Cl}(U) \cap V = \emptyset$ and $\text{Cl}(V) \cap U = \emptyset$, so by Lemma 3.22 b), p123, HB, C is not connected for the Euclidean metric.

(d) S , as all sets are d_0 -closed (Appendix 5, p89, HB).

Q 5.

(a) $\alpha = \sqrt{7} - \sqrt{11}$. Therefore:

$$\begin{aligned}\alpha^3 &= (\sqrt{7})^3 + 3(\sqrt{7})^2\sqrt{11} + 3\sqrt{7}(\sqrt{11})^2 + (\sqrt{11})^3 \\ &= 7\sqrt{7} - 21\sqrt{11} + 33\sqrt{7} - 11\sqrt{11} \\ &= 40\sqrt{7} - 32\sqrt{11}.\end{aligned}$$

Hence $a = 40, b = -32$ (both clearly in $\mathbb{Q}$).

(b) Eliminating, we have:

$$\begin{aligned}\alpha^3 - 32\alpha &= 40\sqrt{7} - 32\sqrt{11} - 32(\sqrt{7} - \sqrt{11}) \in \mathbb{Q}(\alpha) \\ &= 8\sqrt{7} \in \mathbb{Q}(\alpha)\end{aligned}$$

$$\begin{aligned}\alpha^3 - 40\alpha &= 40\sqrt{7} - 32\sqrt{11} - 40(\sqrt{7} - \sqrt{11}) \in \mathbb{Q}(\alpha) \\ &= 8\sqrt{11} \in \mathbb{Q}(\alpha)\end{aligned}$$

(c) Since $\alpha \in \mathbb{Q}(\sqrt{7}, \sqrt{11})$
and, from (b) $\mathbb{Q}(\sqrt{7}, \sqrt{11}) \subseteq \mathbb{Q}(\alpha)$
It follows that $\mathbb{Q}(\sqrt{7}, \sqrt{11}) = \mathbb{Q}(\alpha)$.

(d) Since $\sqrt{7} \notin \mathbb{Q}(\sqrt{11})$ and $\sqrt{7}$ is a root of $x^2 - 7$, a quadratic polynomial over $\mathbb{Q}(\sqrt{11})$,
 $[\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q}(\sqrt{7})] = 2$.

By the KLM Theorem (HB p105),

$$\begin{aligned}[\mathbb{Q}(\alpha) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q}(\sqrt{7})] \times [\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] \\ &= 2 \times 2 = 4\end{aligned}$$

(e) $\alpha^2 = 7 - 2\sqrt{7}\sqrt{11} + 11 = 18 - 2\sqrt{77}$
and so $\alpha^2 - 18 = -2\sqrt{77}$ giving $(\alpha^2 - 18)^2 = 308$.
Hence $\alpha^4 - 36\alpha^2 + 324 = 308$. That is $\alpha^4 - 36\alpha^2 + 16 = 0$.

As the minimal polynomial must have degree 4 (from part(d)),
we conclude that $x^4 - 36x^2 + 16$ is the required polynomial.

Q 6.

(a)

(i) When d is the Euclidean metric, then:

Set	d -connected	d -compact	d -complete
A	Yes	No	No
B	No	Yes	Yes

(ii) When d is the discrete metric (d_0), then:

Set	d -connected	d -compact	d -complete
A	No	No	Yes
B	No	Yes	Yes

(b) With respect to the discrete metric:

 A is:

- not connected because it has at least two elements, e.g. 0, 1 are in A (p127, HB),
- not compact as A is not finite (Examples of compact spaces, p132, HB),
- complete, since any set is complete with the discrete metric (Examples of complete and incomplete spaces, p140, HB).

 B is:

- not connected because it has at least two elements, e.g. 0, 2 are in B (p127, HB),
- compact as B is finite (Example 3, p132, HB),
- complete (p140, HB).

Q 7.

G is a group of order 325, and $325 = 5^2 \times 13$.

(a)

(i) Sylow-5 subgroups have order 25 (and Sylow-13 subgroups have order 13).

Given the question, "25" alone gets the mark for (i).

(ii) We use Sylow's First and Second Theorems (Thm. 1.12 HB p. 39)

Let the number of Sylow 5-subgroups of G be n_5 . $n_5 \equiv 1 \pmod{5}$ and n_5 divides 13. As 13 is prime, n_5 is 1 or 13, so $n_5 = 1$. Thus G has a unique Sylow 5-subgroup, which can be written as H_5 .

Given the use of the definite article in the wording of (a)(i) one might argue that we are told that H_5 is unique. I would not advise this.

Similarly, if the number of Sylow 13-subgroups of G is n_{13} then $n_{13} \equiv 1 \pmod{13}$ and n_{13} divides 25, and so must be one of 1, 5, or 25. Thus $n_{13} = 1$ and so G has a unique Sylow 13-subgroup, which can be written as H_{13} .

(iii) By Lemma 1.9 HB p. 39, as the Sylow 5-subgroup and the Sylow 13-subgroup are unique, they are normal.

As G is a group of order 325, and the orders of the Sylow subgroups of G are given by $|H_5| = 25$ and $|H_{13}| = 13$, we have

$$|G| = |H_5| \times |H_{13}|$$

It can also be seen that $|H_{13}|$ and $|H_5|$ are coprime as $\text{hcf}(13, 25) = 1$.

Then by the internal direct product theorem for finite groups (Thm. 2.3, HB p. 39), G is isomorphic to $H_5 \times H_{13}$.

Moreover, H_5 is Abelian as it has order 5^2 (Thm. 2.15, HB p. 36) and H_{13} is Abelian as it has prime order (Thm. 3.1, HB p. 36).

Thus G is a direct product of Abelian groups and so is itself Abelian.

No HB ref, but see Ex. 1.4, Book B, p90.

(iv) The non-isomorphic possibilities for G are $\mathbb{Z}_{13} \times \mathbb{Z}_{25} (\cong \mathbb{Z}_{325})$ and $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{13}$.

(b)

(i) Let G be a non-Abelian group of order p^3 where p is prime, and let $Z(G)$ denote the centre of G . By Proposition 2.9 HB p. 36, $Z(G)$ is a normal subgroup of G .

By Lagrange's Theorem (Groups) (HB p. 26 Theorem 2.16), the order of $|Z(G)|$ must be 1, p , p^2 or p^3 .

If $|Z(G)| = p^3 = |G|$ then $G = Z(G)$, implying that G is Abelian - impossible as G is given to be non-Abelian. Thus $Z(G) \neq p^3$.

By HB p. 36 Theorem 2.13, as G is a group of prime power order it has a non-trivial centre so it follows that $|Z(G)| \neq 1$.

If $|Z(G)| = p^2$, then $|G/Z(G)| = p$ and so $G/Z(G) \cong \mathbb{Z}_p$ (Prop. 3.1, HB p36) which is cyclic. As a consequence, by Thm. 2.14 (HB p36), G would be Abelian - again, impossible. It follows that $|Z(G)| \neq p^2$.

The only remaining possibility is $|Z(G)| = p$ which means that $Z(G) \cong \mathbb{Z}_p$.

- (ii) From the Prime Power Subgroups Theorem (HB p. 39 Theorem 3.1) G has a subgroup of order p^2 .

$Z(G)$ is a normal subgroup of G so by the Correspondence Theorem (Theorem 4.17 HB p. 30) there is a one-one correspondence between the subgroups of G that contain $Z(G)$ and the subgroups of the quotient group $G/Z(G)$, and this correspondence maps normal subgroups of G to normal subgroups of $Z(G)$ and vice versa.

Now $|Z(G)| = p$

So $|G/Z(G)| = p^2$, hence $G/Z(G)$ is Abelian.

$G/Z(G)$ has a subgroup K of order p (by the Prime Power Subgroups Thm., HB, p39), which is normal.

It follows that G has a normal subgroup H by the Correspondence Thm., (HB, p30) which in addition gives (through Cor. 4.18, HB, p30)

$$|H| = \frac{|G||K|}{|G/Z(G)|} = p^2.$$

Jamal Khan has supplied this slick alternative for this part that does not use the Correspondence Theorem. It does not build directly on part (b)(i) and so may be thought to contravene the wording of the question. Neat though.

As G is a p -group of order p^3 , we may apply Theorem 4.3 (HB p. 38) to deduce that there exist normal subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset H_3 = G,$$

where $|H_i| = p^i$ for $i = 0, \dots, 3$. Thus G has a normal subgroup of order p^2 .

Q 8.

$R = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} : a, b \in \mathbb{Z}\}$ and $N(a + b\sqrt{-7}) = a^2 + 7b^2$ is a multiplicative norm on R .

(a) For $r \in R$ then $r = s + t\sqrt{-7}$, where $s, t \in \mathbb{Z}$.

If $N(r) = 1$, then $s^2 + 7t^2 = 1$ for some integers s, t . If $t \neq 0$, then $s^2 + 7t^2 > 1$, and no solutions exist. Thus $t = 0$ and $s^2 = 1 \rightarrow s = \pm 1$. Hence if $N(r) = 1, r = \pm 1$.

If $N(r) = 2$, then $s^2 + 7t^2 = 2$ for some integers s, t . If $t \neq 0$, then $s^2 + 7t^2 > 2$, and no solutions exist. Thus $t = 0$ and $s^2 = 2$. But $s^2 = 2$ has no solution in $\mathbb{Z}$, hence there are no elements r such that $N(r) = 2$.

(b) Assume 2 is reducible in R , then $2 = xy$ for some $x, y \in R$ and by Definition 2.5 of HB p54 neither x or y is a unit in R . Since the norm N is multiplicative $N(2) = N(x)N(y) = 4$. There are two possibilities:

1. Either $N(x) = 1$ and $N(y) = 4$ or $N(x) = 4$ and $N(y) = 1$. Hence one of x or y is a unit and in this case 2 is irreducible in R .
2. $N(x) = N(y) = 2$. From part (a) no element r exists such that $N(r) = 2$. Hence in this case 2 is also irreducible in R .

Assume $1 + \sqrt{-7}$ is reducible in R , then $1 + \sqrt{-7} = xy$ for some $x, y \in R$ and by Defn. 2.5 HB P.54 neither x or y is a unit in R . Since the norm N is multiplicative $N(1 + \sqrt{-7}) = N(x)N(y) = 8$. There are two possibilities:

1. Either $N(x) = 1$ and $N(y) = 8$ or $N(x) = 8$ and $N(y) = 1$. Hence one of x or y is a unit and in this case $1 + \sqrt{-7}$ is irreducible in R .
2. Either $N(x) = 2$ and $N(y) = 4$ or $N(x) = 4$ and $N(y) = 2$. In each case one of $N(x)$ or $N(y) = 2$. From part (a) no element r exists such that $N(r) = 2$. Hence in this case $1 + \sqrt{-7}$ is also irreducible in R .

The exact same argument as used for $1 + \sqrt{-7}$ can be used to show that $1 - \sqrt{-7}$ is irreducible in R .

It follows that all of $2, 1 + \sqrt{-7}, 1 - \sqrt{-7}$ are irreducible in R .

(c) $(1 + \sqrt{-7})(1 - \sqrt{-7}) = 8 = 2 \times 2 \times 2$. By Definition 4.1 of HB p.56, since $1 + \sqrt{-7}$ and $1 - \sqrt{-7}$ do not pair off with 4 and 2 as associates then R is not a UFD.

Let $I = \langle 2, 1 + \sqrt{-7} \rangle$ be the ideal of $\mathbb{Z}[\sqrt{-7}]$ generated by 2 and $1 + \sqrt{-7}$.

(d) If $\alpha + \beta\sqrt{-7} \in I$ then for some $a + b\sqrt{-7}, c + d\sqrt{-7} \in R$, $\alpha + \beta\sqrt{-7} = 2(a + b\sqrt{-7}) + (1 + \sqrt{-7})(c + d\sqrt{-7}) = 2a + c - 7d + (2b + c + d)\sqrt{-7}$.

Hence $\alpha - \beta = 2a - 2b - 8d = 2(a - b - 4d)$ an even number, as required.

(e) If $\alpha, \beta \in \mathbb{Z}$ are both even then $\alpha = 2p$ and $\beta = 2q$, where $p, q \in \mathbb{Z}$, so $\alpha - \beta = 2(p - q)$ is even. Then $\alpha + \beta\sqrt{-7} = 2p + 2q\sqrt{-7} = 2p + 2q(1 + \sqrt{-7}) - 2q$.

Therefore $\alpha + \beta\sqrt{-7} = (p - q)2 + (2q)(1 + \sqrt{-7})$ and so is in I .

If $\alpha, \beta \in \mathbb{Z}$ are both odd let $\alpha = 2p + 1$ and $\beta = 2q + 1$ ($p, q \in \mathbb{Z}$) so $\alpha - \beta = 2(p - q)$ is even. Then $\alpha + \beta\sqrt{-7} = (2p + 1) + (2q + 1)(\sqrt{-7}) = (2p + 1) + (2q + 1)(1 + \sqrt{-7}) - (2q + 1)$.

Therefore $\alpha + \beta\sqrt{-7} = (p - q)2 + (2q + 1)(1 + \sqrt{-7})$ and so is also in I .

Thus

$$\{\alpha + \beta\sqrt{-7} : \alpha, \beta \in \mathbb{Z}, \alpha - \beta \text{ is even}\} \subseteq I$$

As part(d) provides the reverse inclusion, the two sets are equal.

(f) If $x = a + b\sqrt{-7} \in \mathbb{Z}[\sqrt{-7}]$ and $x \notin I$ then by part(e) $a - b$ is odd.

Now $1 + x = (1 + a) + b\sqrt{-7}$ and $(1 + a) - b$ is even and so, again by part(e), $1 + x \in I$.

(g) Suppose that there is another ideal J , with $I \subset J \subseteq R$.

Since I is a proper subset of J , there is $x \in J$, with $x \notin I$. Then by the previous part $1 + x$ belongs to I and so J . But then $1 = (1 + x) - x \in J$, from which it follows that $J = R$.

Thus I is a maximal ideal of R .

And here is another approach to the last part suggested by a Jamal.

$I = \{\alpha + \beta\sqrt{-7} : \alpha, \beta \in \mathbb{Z}, \alpha - \beta \text{ is even}\}$, so I has two cosets, itself and $I + 1 = \{\alpha + \beta\sqrt{-7} : \alpha, \beta \in \mathbb{Z}, \alpha - \beta \text{ is odd}\}$. Then R/I is isomorphic to the field $\mathbb{Z}_2$. Thus I is a maximal ideal by Theorem 4.23 HB p.99.

Q 9.

(a)

- (i) From the given definition, $d(a, b) \geq 0$ for each $a, b \in \mathbb{N}$. If $a = b$, or if $a - b = 0$, then $d(a, b) = 0$ immediately. If $d(a, b) = 0$, then the only possibility is when $a - b = 0$, or when $a = b$ from the definition. (M1) holds.
- (ii) We note that $a - b$ is 0 if and only if $b - a$ is 0 (and so $a - b$ is non-zero if and only if $b - a$ is non-zero). Also $a - b$ is even if and only if $b - a$ is even (and so $a - b$ is odd if and only if $b - a$ is odd). It therefore follows that $d(a, b) = d(b, a)$, for all $a, b \in \mathbb{N}$. That is, (M2) holds.
- (iii) To show (M3) holds, take any $a, b, c \in \mathbb{N}$. We want to show that

$$d(a, c) \leq d(a, b) + d(b, c) \text{ for all } a, b, c \in \mathbb{N}$$

Consider cases.

- (1) if $a = c$, the LHS = 0 and so the result follows easily, as RHS ≥ 0 .
 - (2) if $a = b$, the LHS = RHS and again the result follows easily. Similarly, if $b = c$.
 - (3) Thus we are reduced to the case where a, b, c are all distinct. But in this case, the LHS is at most 2 while the RHS is at least 2, and the result follows. Thus (M3) holds.
- (b) Let (a_i) be an eventually constant sequence in $\mathbb{Z}$, then there is $N \in \mathbb{N}$ such that $a_n = a_m (= a)$ for all $m, n > N$ (Defn. 3.4 HB p 73).
Thus $d(a_m, a_n) = 0$ for all $m, n > N$.
Since $(\mathbb{Z}, d)$ is a metric space it follows immediately from Definition 1.3 (HB p 135) that (a_i) is d -Cauchy.
- (c) Let (a_i) be a d -Cauchy sequence in the metric space $(\mathbb{Z}, d)$ then by Defn. 1.3 (HB p135) for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(a_m, a_n) < \varepsilon$ for each $m, n > N$.
From the definition of d , for $\varepsilon \leq 1$ then $d(a_m, a_n) < \varepsilon$ only if $a_n = a_m$. Hence, it follows that if (a_i) is d -Cauchy then it is eventually constant.
- (d) Since every d -Cauchy sequence in $(\mathbb{Z}, d)$ is eventually constant by part (c) then it is d -convergent (Defn. 3.4 HB p73) and hence $(\mathbb{Z}, d)$ is a complete metric space by Definition 2.1(HB p136).